Notes on 2.3 (Newton's method)

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Newton's Method

Description:
another root finding algorithm for a function $f(x)$

Algorithm:
let p_0 be a guess.
let $p_i = p_{i-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$

Intuition

pretend f was linear:
$f(x) = f(p_0) + f'(p_0)(x - p_0)$

solve for $f(x) = 0$
get $x = -\frac{f(p_0)}{f'(p_0)} + p_0$

If this didn't work we pretend it is linear from this new point.

Theorem 2.6
If $f \in C^2([a,b])$, $f(p) = 0$, $f'(p) \neq 0$, then there exists $\delta > 0$ such that if $p_0 \in [p - \delta, p + \delta)$, then Newton's method converges.

Note: this theorem is not very helpful.

proof of theorem 2.6

1. let $g(x) = x - \frac{f(x)}{f'(x)}$, so we are now just studying iterations of $g(x)$
2. need to check conditions of fixed point convergence of g
    a. $f'(p) \neq 0$, $f' \in C^1$, therefore near p, $f'(x) \neq 0$, therefore we aren't dividing by 0, so $g \in C^0$ near p
    b. $g'(x) = 1 - \frac{f'(x)^2 - f(x)f''(x)}{f'(x)^2} = \frac{f(x)f''(x)}{f'(x)^2}$, therefore $g \in C^1$ near p and $g'(p) = 0$, therefore g' is small near p
    c. since g' is small near p, then g maps an interval into itself (think about how the graph is flat)
3. apply theorem about fixed point convergence of g.

Secant method
a variation of Newton's method that doesn't involve computing derivatives of f.

Algorithm
begin with two guesses p_0, and p_1

then $p_n = p_{n-1} - \frac{f(p_{n-1})(p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})}$

Intution
just replace derivative with discrete approximation of derivative.